

Math 213W
Writer's Manual

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A) Introduction

Writing a paper in mathematics is probably a new experience for you. It has much in common with writing papers in your other subjects. For example, all your papers should avoid mistakes in spelling and grammar and your conclusions should be well supported by your arguments. However, each subject area has its own special needs and requirements. This manual should be of help to you in meeting these needs and requirements in the field of mathematics. It is not meant to replace the writing handbook you use in your writing courses but to supplement it.

Throughout this manual many of the suggestions given will be illustrated with the paper, *Playing Games with Mathematica*, by Cowen and Dickau, which appeared in the journal *Mathematica in Education and Research* in 1998 [Volume 7, Number 3, pp.5-10]. A complete copy of the paper can be found in the appendix. We shall refer to this paper as the "Games paper." Also the text for the course, "Discovering Mathematics with Mathematica," (Cowen, R. and Kennedy, J., Erudition Press, 2001) also provides a model of mathematics writing based on Mathematica investigations.

In order to introduce you to scholarly writing in the field of mathematics, we will assume that the paper you produce is to be submitted to a math journal and advise you accordingly. Then, if your results are outstanding and you write them up well, your paper will be ready for submission to a journal; if not, you will learn this format for the future. In fact if there is sufficient interest we could produce our own journal from the articles produced in this class! In any case, in addition to improving your mathematics writing, this should make you a more thoughtful reader of mathematics articles.

B) Finding a Problem

By the time you are ready to write your paper, you will have already seen *Mathematica* investigations of many problems. Perhaps you have found some of these problems especially appealing and have questions of your own about them that have not yet been fully answered. Those unanswered questions can often form the basis for a paper, or sometimes, one of the problems from the text can serve as a basis for a successful research project.

Also it is often possible to take a successful investigation and vary the conditions slightly thereby getting a new problem to investigate. In this case many of the methods used in the original problem can often be adapted successfully as well. In our study of games and puzzles in our text, even a slight variation in the rules can change the outcome and thus produce a new problem. For example, in Chapter 8, the Lights Out Puzzle is studied and, as mentioned, a solution exists for any size board. The rule in Lights Out is: any square that is touched changes the state of that square and that of its neighbors (those with which it shares an edge). Now suppose just the neighbors change their state, but not the square itself; we then have a new puzzle, which has been called "Orbix." It is no longer the case for the Orbix puzzle that any size board has a solution. The question then is: which ones do and which ones don't have solutions? This turns out to be a fruitful problem to investigate with *Mathematica* and, although a lot has been discovered so far, it is still a long way from being completely solved!

Another example is in the game of Cram where pieces are dominos, each of which cover two squares of a rectangular board. Suppose instead, these dominos covered three squares in a line or perhaps were L shaped, etc. [see the Final Remarks section of the Games paper for more on this]. Discovering which player will win on various size boards with these altered rules can also make a good research project.

Students have also gotten very good papers by varying the $3x+1$ problem.---say by instead using $3x+5$, or $3x+7$, etc. Finally, if you cannot find a problem that you are happy with ask your teacher for suggestions.

Once you think you have a topic, you should write a one page proposal which expresses clearly what you will be investigating and what methods you will use. Writing proposals is a common experience for most researchers; often funding depends on a proposed course of research. So make sure your proposal is well written. Your teacher will return this proposal to you with comments on whether your project has a good chance of success, along with suggestions you might want to consider while conducting your research. In addition your proposal will receive a grade that reflects the care you put into constructing the project and how well you have communicated your ideas for your project.

C) Conducting the Research.

The research for your paper can be carried out in the Lab (The hours that the Lab will be open during the semester will be posted.). Also there are other computer labs on campus which have Mathematica available; the City University has a site license for Mathematica and can put it on any computer in the University. Also you can run your own copy of Mathematica at home through this site license. Ask your instructor how to obtain your copy of the software. The advantage of doing your research in our Lab is that the Lab Assistant is familiar with Mathematica can help you if you get stuck. Also your instructor can help you during office hours. Some class sessions will be allotted to research for the purpose of providing help to you, as well. Don't hesitate to ask for help. It is only rarely that a student will not need any help at all.

What might you need help for? Here are three situations in which students have asked for help in the past. There are many other situations where help was requested as well.

1. You may not be able to get anywhere with your original problem and could use suggestions on what to do next.

2. You know what you want to have the computer investigate but you are unable to write correct programs that do what you want.

3. You have produced very interesting data supporting your conjectures and you are now positive they are true, but you have no idea on how to prove your conjectures.

D) Writing the Paper.

A Noble prize winning physicist once joked that there are two kinds of math papers. "In the first kind of paper, you cannot get past the first page. On the other hand, in the second kind of paper, you cannot get past the first paragraph!" Although this criticism is often justified (and applies also to physics papers), we believe that it is possible to write decent, readable mathematics papers; however this takes effort. The purpose of this manual is to assist you in that effort.

1. Deciding your audience.

The same research can often be used in a variety of papers geared to different audiences. You must decide who your intended audience is, how much they know about the topic of your investigation, how familiar they are with *Mathematica* programming, etc. For the Games paper, the audience was the readers of a journal devoted to *Mathematica*, so their familiarity with this programming language could be assumed. (This does not mean, however, that they were all experts.) A good choice of audience for your paper would be your fellow students in this class; they are familiar with basic *Mathematica* programming, but probably not familiar with your particular topic. This is an especially good choice since you must also give an oral presentation to your classmates.

2. Some general writing principles for mathematics papers.

The paper should have a logical structure that seems to you to be natural given the research you have conducted. It is a good idea to

write an outline to impose this structure. The structure you have chosen should then also feel natural to your readers. To achieve this you must also pay attention to transitions. The reader should feel that each sentence follows naturally from the one before; the same goes for paragraphs. This natural flow will make it easier for the reader to understand your work.

Try to be clear and precise in your language. Good writing takes work. Often it takes several revisions before the paper is ready. When you revise, try to find the best possible way to express each sentence. Many people suggest reading your work out loud to someone else or to yourself to catch confusing expressions. Keep your sentences short and to the point. Again, assume you are explaining to a fellow student.

3. Numbering schemes.

Sections should be numbered consecutively, starting with 1, for the Introduction (see below). Inside each section, number each result or comment that you may want to refer to later. For example, you can number theorems as Theorem 1, Theorem 2, etc., or you can use your section numbers as well to number things, for example, Theorem 2.1 (the first numbered item in section 2), Corollary 2.2 (the second numbered item in section 2), Definition 2.3, Remark 2.4, etc. This does not mean that your paper should mainly be a list of numbered items--quite the contrary. These items can perhaps be considered the bare bones of the paper; it is, however, the "connective tissue" which contains the "meat" of the paper and makes it informative to the reader. These connective parts provide the narration that tells how the results obtained fit together to form a coherent whole.

4. Choice of Title

A title should, generally, tell potential readers whether the paper will be of interest to them and it should be short. These two criteria can be somewhat contradictory; therefore a compromise must often be made. For example, "An investigation of the $3x + 17$ problem" is not

very informative because no one will recognize this problem; for, although the $3x+1$ problem is fairly well known, " $3x+17$ " is not. On the other hand, "Investigating the variation of the $3x+1$ problem which replaces $3x+1$ by $3x+17$," is informative but probably too long. A compromise might be, " $3x+17$, a variation of the $3x+1$ problem." The title of the Games paper, "Playing Games with *Mathematica*," is accurate and cute and hopefully will attract readers.

5. The Abstract

Many journals follow the title by an abstract that makes up for the brevity of the title. Its purpose is announce new results and also to allow the reader to make an informed decision about whether to spend time and effort understanding this work. This section is generally short and to the point.

Mathematica in Education and Research did not use abstracts, probably in an attempt to appear less formal and more "user-friendly." Similarly, papers for this course do not have to have an abstract. It is up to you.

If you want to include one, your abstract should probably consist of only a few sentences. An abstract is often written after the body of the paper is finished, so you might want to postpone writing it until then.

6. Introducing the Problem.

It is customary to have a section in which you introduce the problem you are investigating. You might also explain how it relates to other problems that have already been investigated and what results you have obtained. Don't worry about repeating material from the Abstract (assuming you have one); in fact you should make it totally independent of the Abstract. You can also talk about the methods you have used to obtain your results in a general way if you think they are interesting--sometimes a novel method developed to solve a problem is more interesting than the problem itself!

You might then outline the rest of the paper telling what will be found where (that is, in which section). Everything here should be clearly laid out with as little jargon as possible.

Keep in mind your intended audience and what they already can be assumed to know. For example, the readership of *Mathematica in Education and Research* can be assumed to know a lot about *Mathematica* programming, but not much about impartial games, and so we wrote our introductory paragraph accordingly.

Much can be learned by playing games. The beginnings of probability and statistics derive from questions asked by gamblers from their experiences at the gaming tables. Here we consider games where reasoning, not chance, are paramount. In particular, we consider the class of so-called impartial games, defined below. The basic mathematical theory of such games is easily explained and there is an algorithm, known as the Sprague-Grundy algorithm, which enables a player with sufficient computational resources to win, when confronted with a winnable position. Many interesting questions can be asked and answered by students, at all levels, about particular impartial games. We will show how *Mathematica* can be used to play various games and to investigate which player will win (assuming proper play). In this way, we can make and test conjectures, and can discover and prove new theorems. All this while playing games!

If the paper were intended for a journal specializing in these kinds of games, the Introduction, as well as the rest of the paper, would have to be very different! For example, if the paper were to be written for a journal specializing in games, we might write the Introduction something like this.

We investigate the impartial game, Cram. Cram is played on a rectangular board ruled into squares. Players take turns placing dominos on unoccupied squares and the last player to place a domino is the winner. The game has been called various names, including "Plugg" and "Dots and Pairs" and has been studied by various authors [Berlekamp, 1985]; Gardner, 1974]. We show that the second player wins, with proper play, on the 5 by 5 board; this solves an open problem in Gardner. Our proof consists of computing the Grundy number of the board using the *Mathematica* programming language; it is, however, necessary to break up the computation using various symmetry arguments.

7. Division into sections

A journal article is usually divided into sections. Give a lot of thought to what these sections should be; ideally, they will give your work a logical structure that will help you and the reader understand what you want to convey. It can also point out to you places in which there are gaps in your reasoning and hence where more work is needed. The Games paper, although short, is still broken up into 7 sections: Introduction, Impartial Games, Programming the Sprague-Grundy algorithm, Visualization, Large Computations, Final Remarks, and References. Some articles follow the Introduction with a section variously called "Preliminaries," or "Background," etc., which formally introduces terminology that will be used throughout the paper. Others introduce terminology on the fly; that is, as needed. We have several copies of the journal, "The American Mathematical Monthly" around in the Lab and you are welcome to look at the articles in this journal to see various ways articles have been divided into sections, etc.

8. Making Definitions

In the body of the paper, you can define things properly. Avoid long lists of definitions and don't bother defining concepts your reader will already know. Deciding what to define and how general to make your definitions is an art that is learned by experience and reading good papers. It often happens that making the right definition leads to a breakthrough. A good definition points out a key concept that you think is important for your work and could be important in the future. For example, the definition of pi, as the ratio of the length of the circumference of a circle to that of its diameter, has enriched mathematics for thousands of years; so give some thought to your definitions. In the paper you will often define concepts that are already known in the literature but may not be known to all your readers; these can be defined in the body of the text, using italics for the defined concept. For example: "an *impartial* game is a two person game where each player has the same set of moves." If, however, you are making a new definition that will play a key role in your paper, you may want to define it more formally, as Definition 2.4 (4th item in section 2), say, so that you can also refer easily to this definition in what follows.

9. Computer methods and data

You should also tell what methods you are using to obtain your results, what computer programs you are using, and if they are short, include them in the section; otherwise put them in an appendix. Also tell how your programs were obtained--that is, are they simple modifications of programs used in the course or are they original programs you created from scratch to meet a special need. Explain what data you have obtained with the programs, how you are running the programs, and also any tests you have run to ensure that the programs are correct--if this is an issue.

Be sure to present your data as clearly as possible, whether in tabular form or as a graph, making sure to label axes or table entries, and then carefully explain how your data supports your conclusions. Examples of good diagrams and tables can be found in your textbook, in the Mathematica manual (included with the software) and in journals available in the Lab

10. Conjectures and Proofs

It is often possible to make conjectures based on evidence obtained from computer data. For example, for the Orbix puzzle, two students, in two different classes came up with the same conjecture, that the n by n square grids with n even, always have a solution, while if n is odd, there is never a solution. They based this on their computer solutions for even boards up to 100 by 100, along with the computer indication of the lack of a solution when n is odd. This is a bold conjecture, but seemed warranted from the evidence obtained. In fact, I, together with some colleagues at Queens, have been able to prove this conjecture, so that now it is a Theorem.

Even if you cannot prove your entire conjecture it may be possible to prove parts of it. For example, if you have a conjecture about all rectangular grids that you are unable to prove, perhaps you can prove it for all grids of width 2 or 3, etc. It often helps to try these small cases by hand rather than with the computer.

If you verify with the computer a large number of special cases of a general conjecture, this should be reported, together with how long it took your programs to run on the computer you used.

Try to find ways to run more or bigger cases by improving your algorithm or programming, using symmetry arguments (see the Games paper for an example of this) etc. It is interesting that the $3x+1$ conjecture has now been verified for extremely large numbers.

11. The Conclusion

It is desirable to have a final section that can be called "Conclusion" or "Final Remarks." Here is where you sum up your major accomplishments in the paper and suggest areas that might be fruitfully investigated, by you or others, in the future. You can also thank anyone who has provided significant help to you in the preparation of your paper.

Things that do not belong in the Conclusion include your enjoyment or lack of enjoyment in producing this work, your personal feelings about the material, etc. This is a scientific work and you are not the subject of this inquiry.

12. References

Any time you use material from another source in your paper, you must give a reference to that source. This is done by explicitly stating the nature of your indebtedness in the text followed by a number in brackets that is the number of the source in the list of references provided by you at the end of your paper. In addition, references are often used to inform the reader where more material on a certain topic can be found.

References should be listed in alphabetical order using the last names of the first listed author to determine order. For example, a journal article might be listed as:

[1] Bartels, J., Smyth, P., A decision procedure for qualification theory, *J. Local and Global Analysis*, 12(1999), 14 - 23.

Here 12 is the volume number, 1999 is the year of publication, and 14 - 23 are the pages in which the article appears.

A book might be referenced as:

[2] Davidson, M., *Recursive Algorithms*, Cambridge University Press, Cambridge, England, 2002.

A source you found on the Internet is cited similarly with the addition of the words, "online," and date of your visit to the site, as well as the URL.

Different math publications can have slightly different ways of listing references. Choose one format and follow it. For examples, look at the *American Mathematical Monthly* or any math journal in the QC Library.

13. Typing the paper

There are many choices for typing your paper. Assuming you are going to use a computer to word process, *Mathematica* itself is quite capable as a word processing program. The Games paper was produced with *Mathematica*. In *Mathematica* each cell can be designated as an input or text cell. The default is input. To change to text, use the pull-down Format menu; select "style" and then "text." The Format menu can also be used to change the font, face (bold, underline), size of font, justification, etc. Math symbols are available using the Palettes (on the File menu).

Microsoft Word can also be used to produce the paper. Many math symbols are available (pull down the Insert menu and select "symbol") and programs and diagrams can be exported from *Mathematica*. Diagrams and programs from *Mathematica* can be copied and pasted into your Word document; use "Copy As" and select "PICT" in

Mathematica and then use "Paste" from the Edit menu in Word.

Many mathematicians use Tex to produce professional quality documents; however, unless you are very familiar with this method of document preparation it is best to use *Mathematica* or a standard word processing program, such as Word.

14. Submitting your paper. Your paper is due on the day the final examination for the course is scheduled. If you want it returned with comments from your instructor, enclose a stamped self-addressed envelope with sufficient postage as well as a second copy of your paper. Your instructor will retain one copy.

E) The Oral Presentation.

In addition to submitting the written paper you will have to make an oral presentation of your results to the class. Your presentation will receive a grade and will be counted toward your final grade.

Mathematicians often attend conferences where they give oral presentations of results they have obtained to other interested mathematicians. These presentations can range in duration from 10 minutes to an hour. You will have 15 minutes for your presentation. Of course one usually cannot present a full paper in such a short time, so the problem is what to present and how to present it. In general it is a bad idea to simply project to the audience pages of your written paper. Instead you should try to convey your most important findings. These should be motivated and explained.

Don't put too much on any one slide (projected page of your presentation). Use type big enough to be seen at the rear of the room. It is best to just outline your remarks on each slide, and not to read verbatim from the slide. Any visual material you can include is welcome!

You should practice giving your talk several times before you give it to

the class. See that it doesn't need more time than allotted. We will set aside one class for practicing your talk. Students will break up into groups of about three and give their talks to each other. This can be done in clusters using the computers around the room and also it will be possible to project a few slides from the front of the room to see how they will appear to your audience. You will give and receive advice that, should result in a better presentation. Sheets will be provided to guide you in giving advice. These will consist of a list of criteria on which a talk can be evaluated.

Slides can be prepared using Mathematica. In fact Mathematica 5 makes special provisions for creating slide shows. See that help browser for more on this. It is even possible to give a dynamic presentation using Mathematica in that you could run programs during your talk to show results obtained. However it is often better to just show the output previously obtained because sometimes things don't run as you intended on a different computer and this can be embarrassing. Alternatively a presentation can be assembled using software such as PowerPoint if you are familiar with it.